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The far field of high frequency convected singularities in sheared flows, with an application to jet-noise prediction

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The purpose of the present paper is to derive expressions for the pressure fields of various high frequency convected singularities immersed in a unidirectional sheared flow. These expressions include the simultaneous effects of fluid and source convection and refraction. These results are then combined to predict the far-field directivity of cold round jets. It is found that the agreement between experiment and the present theory is quite good at a source Strouhal number of unity but that this agreement deteriorates as the source frequency is increased. Our theoretical results show the explicit form of the 'refraction integral' and that convective amplification for the pressure of a quadrupole is increased by a factor of $(1-M_J \cos \Theta)^{-1}$ over the classical results, where M_J is the jet Mach number and Θ is the angle from the jet axis. Thus acoustic/mean-flow interaction not only implies refraction but also additional convective amplification due *not* to source convection but to fluid motion.

1. Introduction

Lighthill (1952), in his classic theory of jet noise, identified the most prominent source of noise as the double divergence of the tensor **uu**, where **u** is the fluid velocity. He also showed that the acoustic pressure fluctuations that are driven by this source obey the classical wave equation. Since the source of noise is embedded in the jet, the pressure fluctuations propagate through a region of non-uniform velocity (and perhaps temperature) before they reach the observer. The Lighthill theory clearly fails to account for this physical effect; that is, it does not take *acoustic/mean-flow interactions* into account *explicitly*.

Recently it has been recognized, especially through the work of Mani (1972, 1973, 1976*a*, *b*), that these acoustic/mean-flow interactions are extremely important and explain quantitatively many of the observed characteristics of the noise of cold and hot jets. Perhaps the most significant finding of Mani is that 'convection amplification' is frequency dependent, where in the definition of convection amplification we now include both source and fluid convection effects (i.e. a non-zero jet velocity). Of course, several other authors, notably Ribner (1962), Csanady (1966), Schubert (1972) and Pao (1973), have qualitatively explained a number of phenomena through acoustic/mean-flow interactions.

The starting point of Mani's theory is Lilley's (1972) equation. Considerable mathematical complexity can be eliminated, as was done by Mani, by assuming that the jet velocity and temperature profiles can be represented sufficiently accurately by constant (i.e. slug) profiles. Indeed this assumption is surprisingly good for both hot and cold jets except at high frequencies, where Mani's theory *systematically* fails. The purpose of this paper is to bridge this gap and to provide explicit asymptotic results at *high* frequencies for the pressure fields of various convected singularities. Using a theory of Ribner (1969), the acoustic fields of cold or hot jets can be readily constructed from these pressure distributions (see also Mani 1976*a*, *b*).

The starting point of our theory is also Lilley's equation, in which the jet velocity and temperature profiles are left as arbitrary functions of the radial variable r. The relevance of Lilley's equation to jet noise has been questioned by a number of authors for various mathematical and physical reasons. We feel that if acoustic/mean-flow interactions are important, as they indeed are, the Lilley equation must be a *first* approximation of these effects. This conjecture is supported by the success of the work of Mani and the most recent publication of Ffowcs Williams (1974).

Thus the present work is quite reminiscent of the high frequency work of Pao (1973). However, there are a number of important differences, which we now point out. First, our starting point, as mentioned before, is Lilley's rather than Phillips' (1960) equation. Second, we consider a cylindrical jet rather than a planar shear layer and solve *explicitly* for the pressure fields of various convected singularities. Third, this paper is not concerned with the description of the noise sources themselves: such a description has been given by Proudman (1952) and Ribner (1969) and provides the quadrupole weighting factors for our acoustic theory. These weighting factors tell us how the quadrupoles of various orientations must be combined to produce an 'eddy of convected and isotropic turbulence'. (Since the weighting factors are independent of the angle from the jet axis, we can obtain these factors at $\Theta = 90°$, where acoustic/mean-flow interactions are usually small, so that Lighthill's and Lilley's equations yield nearly identical results.) Fourth, our final expressions for the pure convective amplification (i.e. for the index β of $(1-M_c\cos\Theta)^{\beta}$) differ somewhat from those of Pao (1973) for his so-called $S0$ mode (see Pao's equation (70)).

Our approach is to solve Lilley's equation for a convected point source of circular frequency ω. This solution is a Green's function. The approximate solution that we present is valid to lowest order as $ka \to \infty$ ($k = \omega/c_\infty$), where a is the jet radius and c_∞ is the ambient speed of sound. We also require that $a^n d^n U/dr^n = O(U(0))$ (where $n = 1, 2$ and $U(r)$ denotes the jet velocity); that is, the scale of the shear layer is of order a. This assumption is clearly violated in the immediate vicinity of the jet exit. Similar remarks hold for the jet temperature or acoustic speed $c = c(r)$. From experience with problems of this kind, we know that the above asymptotic theory should be quite accurate for $ka \geqslant 2$, that is for frequencies greater than the peak frequency of jet noise (the exact values of the frequency depend on the jet Mach number). We next show how to obtain the corresponding results for dipole and quadrupole singularities. Finally comparisons with experimental data are given.

We remark that the low frequency end of the jet-noise spectrum can be predicted quite well by another asymptotic theory (Balsa 1975) or by the slug-flow model of Mani (1976*a*, *b*).

2. Formulation of the problem

Assume that our physical space is spanned by a stationary cylindrical coordinate system (r, θ, x'), where x' is along the jet axis. Lilley's equation is

$$L(p; U, x') = \frac{1}{c^2} D_U^3 p - D_U \Delta p - \frac{d}{dr}(\log c^2)\, D_U \frac{\partial p}{\partial r} + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial x' \partial r}$$

$$= \rho D_U \nabla . \nabla . (\mathbf{u}'\mathbf{u}' - \overline{\mathbf{u}'\mathbf{u}'}) - 2\rho \frac{dU}{dr}\frac{\partial}{\partial x'} \nabla . (u_r' \mathbf{u}' - \overline{u_r' \mathbf{u}'}), \tag{1a}$$

with

$$D_U = \partial/\partial t + U \partial/\partial x' \tag{1b}$$

and

$$\Delta = \frac{\partial^2}{\partial (x')^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{1c}$$

where t denotes time, p is the acoustic pressure, $c = c(r)$ is the undisturbed speed of sound, $U = U(r)$ is the mean or time-average jet velocity and $\rho = \rho(r)$ is the mean jet density. The turbulent velocity fluctuations are given by $\mathbf{u}'$ and an overbar denotes an appropriate average (u_r' is the radial component).

If c_∞ is the ambient speed of sound, we require that

$$1 \leqslant c/c_\infty \leqslant c_J/c_\infty, \quad 0 \leqslant U/c_\infty \leqslant M_J, \quad \lim_{r \to \infty} U(r) = 0, \tag{2a–c}$$

where M_J and c_J/c_∞ are given numbers. The solution to (1a) can be written down formally as $t \to \infty$ provided that the solution to

$$L(G; U, x') = e^{-i\omega t}\, \delta(x' - U_c t)\, \delta(r - r_0)\, \delta(\theta - \theta_0)/r \tag{3a}$$

is known. In (3a), ω, $U_c > 0$, and r_0 and θ_0 are given constants (i.e. independent t, r, θ and x'). Of course, (3a) simply defines a Green's function. Actually, using the Galilean transformation $x = x' - U_c t$, it is possible to rewrite (3a) as

$$L(G; V, x) = e^{-i\omega t} \delta(x)\, \delta(r - r_0)\, \delta(\theta - \theta_0)/r, \tag{3b}$$

where

$$V = U - U_c. \tag{3c}$$

Thus the canonical problem that we solve is (3b) with a suitable radiation condition as $(r^2 + x^2)^{\frac{1}{2}} \to \infty$. The solution to (3$a$), or (3$b$), represents the pressure field of a convected monopole source. This pressure field obeys Lilley's equation.

After using the sequence of Fourier transformations,

$$\bar{G} = \frac{e^{i\omega t}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-isx}\, dx \int_{-\pi}^{\pi} e^{in\theta} G\, d\theta, \quad -\infty \leqslant s \leqslant \infty, \quad n = 0, \pm 1, \ldots, \tag{4a}$$

we find that (3b) reduces to

$$\frac{d^2 \bar{G}}{dr^2} + \left[\frac{1}{r} + \frac{d}{dr}\log\left(\frac{c}{c_\infty}\right)^2 + \frac{2s}{k - Ns}\frac{dN}{dr}\right]\frac{d\bar{G}}{dr}$$

$$+ \left[\frac{(k - Ns)^2}{(c/c_\infty)^2} - s^2 - \frac{n^2}{r^2}\right]\bar{G} = \frac{i}{(2\pi)^{\frac{1}{2}}}\frac{1}{c_\infty}\frac{1}{(-k + Ns)} e^{in\theta_0}\frac{\delta(r - r_0)}{r}, \tag{4b}$$

where $N = V/c_\infty$ and $k = \omega/c_\infty$. Note that the inverse Fourier transformation corresponding to (4*a*) is given by

$$G = \frac{e^{-i\omega t}}{(2\pi)^{\frac{3}{2}}} \sum_{n=-\infty}^{\infty} e^{-in\theta} \int_{-\infty}^{\infty} e^{isx}\, \bar{G}\, ds. \tag{5}$$

The coefficient of $d\bar{G}/dr$ in (4*b*) can be eliminated by a standard transformation. If we define

$$P = r^{\frac{1}{2}} \frac{c}{c_\infty} \frac{1}{-k+Ns} \bar{G} \tag{6a}$$

then (4*b*) reduces to

$$P_{rr} + \left\{k^2\left[\frac{(1-N\sigma)^2}{(c/c_\infty)^2} - \sigma^2\right] - \frac{n^2-\frac{1}{4}}{r^2} + \frac{1}{r}\frac{\psi_r}{\psi} + \frac{\psi_{rr}}{\psi} - 2\left(\frac{\psi_r}{\psi}\right)^2\right\} P$$
$$= \frac{i}{(2\pi)^{\frac{3}{2}}} \frac{c}{c_\infty^2} \frac{1}{k^2} \frac{1}{(1-N\sigma)^2} e^{in\theta_0} \frac{\delta(r-r_0)}{r^{\frac{1}{2}}}, \tag{6b}$$

where $\sigma = s/k$ and
$$\psi = \frac{1-N\sigma}{c/c_\infty}. \tag{6c}$$

Let us now observe from (6*b*) that for small values of r the term $(n^2-\frac{1}{4})r^{-2}$ dominates the left-hand side (note that $r^{-1}\psi_r/\psi$ has a removable singularity at $r = 0$), whereas, for $kr \gg 1$ the term $k^2[\ldots]$ dominates since the terms involving ψ are $O(a^{-2})$ by hypothesis and $ka \gg 1$. Thus the terms involving ψ can be neglected for all values of r as long as $ka \gg 1$; therefore (6*b*) simplifies to

$$P_{rr} + \left\{k^2 g^2(r;\sigma) - \frac{n^2-\frac{1}{4}}{r^2}\right\} P = \frac{i}{(2\pi)^{\frac{3}{2}}} \frac{c}{c_\infty^2} \frac{1}{k^2} \frac{1}{(1-N\sigma)^2} e^{in\theta_0} \frac{\delta(r-r_0)}{r^{\frac{1}{2}}}, \tag{7a}$$

where
$$g^2(r;\sigma) = \frac{(1-N\sigma)^2}{(c/c_\infty)^2} - \sigma^2. \tag{7b}$$

The qualitative behaviour of P depends on the sign of g^2: P is 'oscillatory' for $g^2 > 0$ and 'exponential' for $g^2 < 0$. We now focus our attention on the turning points of g^2, that is, on the values of r for which $g = 0$.

3. The turning points

Readers familiar with the classical WKBJ† technique (Morse & Feshbach 1953) may at this point feel that the turning points of (7*a*) are determined by the quantity in the curly brackets rather than g^2 alone. Indeed this is one approach that coincides with the classical results of Langer (see Erdélyi 1956, p. 78). This approach leads, however, to undue complications since the turning points will depend parametrically on n. Our approach is based on the observation that if g^2 were constant the solutions of (7*a*) would essentially be given by Bessel functions for $g^2 > 0$ and by modified Bessel functions for $g^2 < 0$. The turning points, that is the places where we switch from Bessel to modified Bessel functions,

† Perhaps more correctly called the Liouville–Green–Rayleigh–WKBJ technique.

are determined by g^2 alone, independently of n. We shall come back to this point later when we discuss the solutions of (7*a*).

We now examine the turning points of g^2. Let us observe that g^2 is a quadratic function of N with $g^2 \to +\infty$ as $|N| \to \infty$. The N intercepts of this quadratic are given by

$$N_1 = 1/\sigma - c/c_\infty, \quad N_2 = 1/\sigma + c/c_\infty, \tag{8a, b}$$

with $N_2 > N_1$. By hypothesis [see (2*b*)]

$$-M_c \leqslant N \leqslant (M_J - M_c), \tag{8c}$$

where $M_c = U_c/c_\infty$ and, as before, $N = V/c_\infty = (U - U_c)/c_\infty$. Without any loss of generality and without violating any of our previous results we may set M_J and c_J equal to the maximum values of the jet Mach number and acoustic speed: this has been anticipated by the notation.

The assumption that $0 \leqslant M_J - M_c \leqslant 1$, which we now invoke, is not terribly restrictive for jet noise. A more restrictive assumption on our theory is that $M_c < 1$. The latter assumption must be made to guarantee that the particular mathematical procedure that we follow, namely the evaluation of the s integral in (5) by the method of stationary phase, be meaningful. It is, however, possible to extend our results to $M_c > 1$ by the methods used by Seckler & Keller (1959) in connexion with another problem. Of course, we still get a singularity when $M_c \approx 1$ but this singularity can be 'removed' by the techniques developed by Ribner (1962) and Ffowcs Williams (1963). Putting all these assumptions together (8*c*) may be rewritten as

$$-1 < -M_c \leqslant N \leqslant M_J - M_c < 1, \tag{9}$$

or in other words $N \in (-1, 1)$.

When $\sigma \geqslant 0$, $N_2 \geqslant 1$, so that g can vanish at most for one value of N in the interval $(-1, 1)$. The values of r for which g vanishes are given by the solution of (8*a*) for r, that is, by the set of r_σ for which

$$N(r_\sigma) = 1/\sigma - c(r_\sigma)/c_\infty, \quad \sigma \geqslant 0. \tag{10}$$

Equation (10) cannot have a solution when $\sigma \notin [(c_J/c_\infty + M_J - M_c)^{-1}, (1 - M_c)^{-1}]$ since the quantity $N + c/c_\infty$ is bounded both from above and below. The last statement can be made considerably sharper if we assume that there exists an r, say r^*, such that $c(r^*) = c_J$ and $U(r^*)/c_\infty = M_J$, in other words that the maximum jet acoustic speed and jet velocity occur, at least once, at the same radial location. (Note that this assumption of the existence of an r^* is generally fulfilled for jets.) Then there exists at least one turning point r_σ whenever

$$(c_J/c_\infty + M_J - M_c)^{-1} \leqslant \sigma \leqslant (1 - M_c)^{-1}.$$

For $0 \leqslant \sigma < (c_J/c_\infty + M_J - M_c)^{-1}$ there are no turning points; in fact, from (7*b*), $g^2 > 0$. Similarly, for $\sigma > (1 - M_c)^{-1}$ we have $g^2 < 0$.

The corresponding conclusions for $\sigma < 0$ can also be obtained from (8*a, b*). We find that there exist $\sigma_1 < \sigma_2 < 0$ such that $g^2 > 0$ for $\sigma \in (\sigma_2, 0)$, g^2 has *at least one* turning point for $\sigma \in [\sigma_1, \sigma_2]$ and $g^2 < 0$ for $\sigma < \sigma_1$. In general, it is not possible to give the number of turning points and the functional dependence of

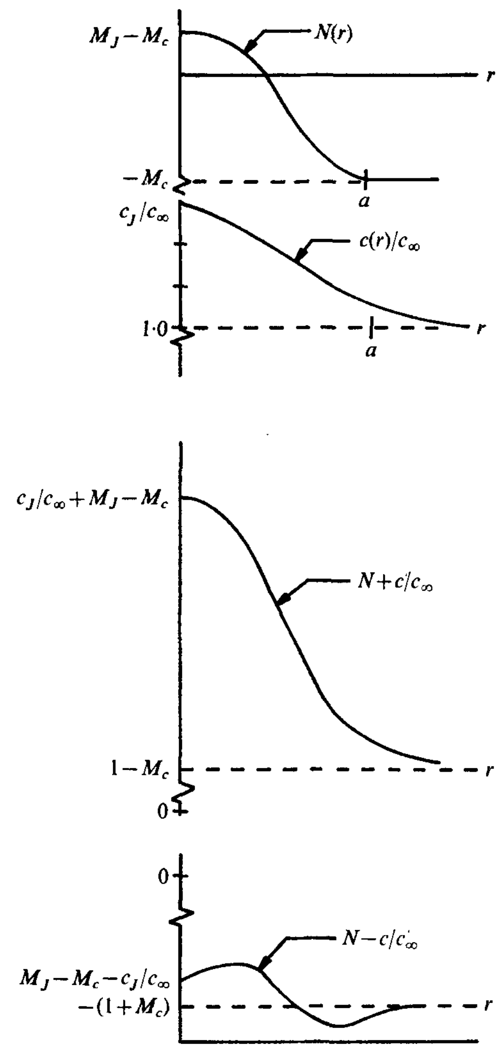

FIGURE 1. Some typical jet velocity and acoustic speed profiles.

σ_1 and σ_2 on M_J, M_c, etc. in closed form. Of course, for *specific* jet velocity and temperature profiles this functional dependence is known; for example, when $c/c_\infty = 1$, $\sigma_1 = -(1-M_J+M_c)^{-1}$ and $\sigma_2 = -(1+M_c)^{-1}$ (see also figure 1).

At this point we jump ahead of ourselves and quote certain results without proof. As stated before, the integral in (5) is evaluated by the method of stationary phase. The point of stationary phase is given by ($\sigma = s/k$)

$$\sigma_* = \cos\Theta/(1-M_c\cos\Theta), \tag{11}$$

where Θ is the angle from the jet axis. Thus for Θ somewhat greater than 90° (i.e. for $\sigma < \sigma_2$) we cannot give the mean-square pressure with any kind of generality, since the number of turning points depends on the shape of the velocity and temperature profiles. This deficiency of the theory is not terribly severe because in jet noise one is almost always interested in the pressure field at the forward angles where the peak noise occurs.

The point that we want to make, in the terminology of Mani, is that there is 'fluid shielding' for $0 \leqslant \Theta < \Theta_c = \cos^{-1}[1/(1+M_J)]$ and generally also for some range of $\Theta > 90°$.

So far we have not placed any real restrictions on $U(r)$ and $c(r)$. We now specify that $dU/dr \leqslant 0$ and $c/c_\infty = 1$. Thus we find that $M_J = U(0)/c_\infty$ and $c_J/c_\infty = 1$. The above profiles approximately describe those of cold round jets. Since U/c_∞ is now monotonic, there exists no turning point for $\Theta_c < \Theta < 180°$ and a *unique* turning point r_σ for $0 \leqslant \Theta \leqslant \Theta_c$. This choice of U and c results in considerable algebraic simplification. In any case, our theory can be easily extended to other velocity and temperature profiles, for example to annular or coaxial jets. In the forward quadrant (i.e. $\Theta \leqslant 90°$) the only difference between the present and the most general theory is in the number of turning points r_σ. We simply assume in this paper that there is at most one r_σ. Those familiar with the WKBJ theory will realize that the treatment of several turning points (especially well separated turning points) affords no great difficulty other than algebraic.

4. The composite solutions

This section is devoted to solving (7a) as $ka \to \infty$ with $c/c_\infty = 1$ and $dU/dr \leqslant 0$. Thus, as discussed before, g^2 has a unique turning point r_σ for $0 \leqslant \Theta \leqslant \Theta_c$. Furthermore, it is possible to show that $g^2 > 0$ for $r > r_\sigma$ and $g^2 < 0$ for $r < r_\sigma$.

In the interest of simplicity we next assume that $r_0 \ll a$, i.e. that the convected singularities are approximately on the axis of the jet. This assumption leads to tremendous mathematical simplification, in so far as the infinite sum in (5) is replaced by a finite sum of a few terms. Mani found that centre-line convected singularities are sufficient for the description of round-jet noise even though the physical source of noise is distributed circumferentially and radially in the shear layer. The experimental results of Atvars *et al.* (1966) show that the net effect of a 'ring' source is very similar to that of a centre-line source, which is an experimental justification for our assumption.

Since at the end of the analysis we set $r_0 = 0$, we may assume that $r_0 < r_\sigma$. Thus, in the vicinity of the jet axis g^2 is negative. We now want to find two special solutions of (7a), one remaining finite as $r \to 0$; the other representing outgoing waves at infinity. These solutions are obtained by the method of matched asymptotic expansions.

We next introduce an inner variable

$$\tilde{r} = kr \tag{12a}$$

and define an inner region such that $\tilde{r}$ is of order unity. In the inner region (7a) reduces to

$$\frac{d^2P^{(i)}}{d\tilde{r}^2} - \left[f^2(0) + \frac{n^2-\frac{1}{4}}{\tilde{r}^2}\right]P^{(i)} = \frac{i}{(2\pi)^{\frac{1}{2}}}\frac{1}{c_\infty}\frac{1}{k^4}\frac{1}{(1-N\sigma)^2}e^{in\theta_0}\frac{\delta(r-r_0)}{r^{\frac{1}{2}}}, \tag{12b}$$

where $P^{(i)}$ denotes the value of P in the inner region as $ka \to \infty$ and

$$f^2(r) = -g^2(r;\sigma). \tag{12c}$$

A solution of (12*b*) that remains finite on $r = 0$ is proportional to

$$\bar{r}^{\frac{1}{2}} I_n[\bar{r} f(0)], \tag{12d}$$

where I_n is a modified Bessel function of order n.

In the outer region, where

$$\bar{r} = r/a \tag{13a}$$

is of order unity, (7*a*) reduces to

$$\frac{d^2 P^{(o)}}{d\bar{r}^2} - (ak)^2 f^2 P^{(o)} = \frac{i}{(2\pi)^{\frac{1}{2}}} \frac{1}{c_\infty} \left(\frac{a}{k}\right)^2 \frac{1}{(1-N\sigma)^2} e^{in\theta_0} \frac{\delta(r-r_0)}{r^{\frac{1}{2}}} \tag{13b}$$

as $ak \to \infty$, where $P^{(o)}$ is the outer solution. The solutions of (13*b*) are given by the WKBJ approximation (Morse & Feshbach 1953) as

$$P^{(o} \sim \frac{1}{f^{\frac{1}{2}}} \exp\left[\pm ak \int_0^{\bar{r}} f \, d\bar{r} \right]. \tag{14}$$

From (14) and (12*d*) we can easily form a composite solution by the method of Van Dyke (1964). This solution is uniformly valid for all $r < r_\sigma$ and remains finite on $r = 0$. The final result is

$$P \sim r^{\frac{1}{2}} I_n[kr f(0)] + \frac{1}{(2\pi k f(r))^{\frac{1}{2}}} \exp\left[k \int_0^r f \, dr \right] - \frac{1}{(2\pi k f(0))^{\frac{1}{2}}} \exp\left[kr f(0) \right]. \tag{15}$$

The most general solution of (7*a*), in the region $r < r_\sigma$, can be written down by inspection from (15) in terms of I_n and K_n and suitable WKBJ solutions, where K_n is a modified Bessel function of the second kind. Note that this general solution is not bounded near $r = 0$.

By a completely similar procedure we can find the composite solution of (7*a*) that represents outgoing waves at infinity. We thus have all the solutions of (7*a*) that are required for the construction of the Green's function.

An implicit assumption† in our definition of the inner region is that the turning point r_σ is 'far enough' outside the inner region. This assumption is clearly violated whenever $\cos \Theta \approx (1 + M_J)^{-1}$, so that our solution *may* fail in this region. The width of this region, that is the 'boundary-layer' thickness, vanishes as $k \to \infty$. The present theory essentially joins the solutions on either side of this boundary layer by a continuous and smooth curve. The experimental data indicate that this is probably a reasonable approach.

A careful mathematical treatment of the present problem is extremely difficult and delicate. For example, when $r_\sigma \approx 0$, the turning point is of second order and the inner solutions are given by $J_{\frac{1}{2}n}$ rather than J_n.

We feel that it is not necessary to consider in detail this boundary layer in order to predict the *overall* features of acoustic/mean-flow interactions.

† The author is grateful to one of the referees for pointing this out.

5. The Green's function

When $r < r_0$, P must be finite, so that in this region P is a constant multiple of (14). On the other hand, for $r_0 < r < r_\sigma$, P is given by the most general solution of (7a), involving I_n, K_n and suitable WKBJ solutions. Finally, when $r > r_\sigma$, P is a constant multiple of the outgoing-wave solution.

Across the source location (i.e. across $r = r_0$) P must be continuous and P_r must jump by an amount that is determined by integrating (7a) across $r = r_0$. Across r_σ, P must satisfy the classical turning-point conditions (Morse & Feshbach 1953). These conditions at r_0 and r_σ determine P uniquely. For $r > r_\sigma$ and $kr \gg 1$ the final result is

$$P = \frac{1}{2c_\infty k^2} \frac{e^{-\frac{1}{4}i\pi}}{(kg(r;\sigma))^{\frac{1}{2}}} \frac{e^{in\theta_0}}{(1-N_0\sigma)^2} \exp\left[-k\int_0^{r_\sigma} f\,dr\right] \times \exp\left[ik\int_{r_\sigma}^{r} g\,dr\right] I_n[kr_0 f(0)], \quad (16)$$

where $N_0 = N(r_0)$.

The transform $\bar{G}$ of the Green's function is given by (6a) as

$$\bar{G} = -kr^{-\frac{1}{2}}(1-N\sigma)P, \quad (17)$$

so that the s integral in (5) necessarily involves the evaluation of an integral of the form

$$\int_{-\infty}^{\infty} \mathscr{F} e^{ikh(\sigma)}\,d\sigma \quad \text{for} \quad k\to\infty, \quad (18a)$$

where

$$h(\sigma) = x\sigma + \int_{r_\sigma}^{r} g(r;\sigma)\,dr. \quad (18b)$$

Under suitable restrictions (we shall in a moment see what these are) it is possible to evaluate (18a) by the method of stationary phase (Carrier, Krook & Pearson 1966, p. 272). If we define

$$x/r = \cot\tilde{\Theta} \quad (19a)$$

it is possible to show that the point of stationary phase of (18a) is given, as $r\to\infty$, by

$$\sigma_* = \frac{1}{1-M_c^2}\left[M_c + \frac{\cos\tilde{\Theta}}{(1-M_c^2\sin^2\tilde{\Theta})^{\frac{1}{2}}}\right]. \quad (19b)$$

In order that the method of stationary phase be applicable, we require that σ_* be real, that is $M_c < 1$. This explains the necessity for the upper limit on M_c. In jet noise it is customary to introduce an angle Θ defined by

$$\cos\tilde{\Theta} = \frac{\cos\Theta - M_c}{(1+M_c^2-2M_c\cos\Theta)^{\frac{1}{2}}}. \quad (20)$$

Note that Θ is the angle between the jet axis and the line connecting the observation point and the source point at the time of emission of the sound wave. In terms of Θ, we find that

$$\sigma_* = \cos\Theta/(1-M_c\cos\Theta), \quad (21)$$

and as $R = r/\sin\Theta \to \infty$, the Green's function (17) becomes

$$G = e^{-i\omega t}\frac{i}{4\pi c_\infty kR}\sum_{n=-\infty}^{\infty}\frac{\exp\left[\mathrm{in}\left(\theta_0-\theta\right)\right]}{(1-M_0\cos\Theta)^2}e^{ikh(\sigma_*)} \times \exp\left[-k\int_0^{r_\sigma}f\,dr\right]I_n[kr_0f(0)], \quad (22)$$

with $M_0 = M(r_0)$ and f evaluated at the point of stationary phase. When there is no turning point (i.e. $\Theta_c < \Theta < 180°$), we set $r_\sigma \equiv 0$ and replace I_n in (22) by $J_n[kr_0g(0)]\exp(-\frac{1}{2}in\pi)$.

Let us remark that the terms in the infinite sum in (22) are even functions of n. Thus (22) can be readily rewritten as a sum of cosines over non-negative integers; this will be done in the following section.

6. The pressure field of convected singularities

Let us begin by seeking the solution of

$$L(\mathscr{S}; U, x') = \rho_\infty D_U e^{-i\omega t}\delta(x'-U_c t)\,\delta(r-r_0)\,\delta(\theta-\theta_0)/r; \quad (23)$$

$\mathscr{S}$ may be thought of as a pressure source. After evaluating the derivatives on the right-hand side, we find that the forcing term in (23) reduces to

$$\rho_\infty[-i\omega e^{-i\omega t}\delta(x'-U_c t)\,\delta(r-r_0)\,\delta(\theta-\theta_0)/r + (U-U_c)\,e^{-i\omega t}\delta'(x'-U_c t)\,\delta(r-r_0)\,\delta(\theta-\theta_0)/r], \quad (24)$$

where δ' is the derivative of the δ-function. Thus $\mathscr{S}$ is expressible in terms of G and $\partial G/\partial x'$. In fact,

$$\mathscr{S} = \rho_\infty[-i\omega + (U_0-U_c)\,ik\sigma_*]\,G, \quad (25a)$$

or

$$\mathscr{S} = -ik\rho_\infty c_\infty\frac{1-M_0\cos\Theta}{1-M_c\cos\Theta}G, \quad (25b)$$

with $M_0 = M(r_0)$ and $U_0 = U(r_0)$.

Now let us observe that the Lilley operator is independent of x', r_0 and θ_0. Thus any derivative (with respect to x', r_0 or θ_0) of the left-hand side of (23) is directly transferable to $\mathscr{S}$. Similarly, when applied to the right-hand side, these differentiations will induce higher-order singularities. These singularities will produce the various dipole and quadrupole solutions of Lilley's equation. For example, a suitable linear combination of $\partial\mathscr{S}/\partial r_0$ and $\partial\mathscr{S}/\partial\theta_0$ produces a transverse dipole, or more precisely, $\partial\mathscr{S}/\partial r_0$ represents an r-dipole (note that we assume, $dM/dr \simeq 0$ at $r = 0$). Using the definitions

$$B_n = \frac{\rho_\infty}{4\pi R}e^{-i\omega t}\frac{\exp\left[-k\int_0^{r_\sigma}f\,dr\right]}{(1-M_0\cos\Theta)(1-M_c\cos\Theta)}\epsilon_n e^{ikh(\sigma_*)}, \quad n = 0, 1, 2, \ldots, \quad (26a)$$

$\mathscr{S}$ can be rewritten as

$$\mathscr{S} = \sum_{n=0}^{\infty}B_n\cos n(\theta-\theta_0)\,I_n[kr_0f(0)], \quad (26b)$$

where $\epsilon_0 = 1$ and $\epsilon_n = 2\,(n \geqslant 1)$. Furthermore, if we set

$$C_n = B_n\,[\tfrac{1}{2}kf(0)]^n/\Gamma(n+1), \quad n = 0,1,2,\ldots, \tag{26c}$$

where Γ denotes the gamma function, the expressions for the *centre-line* convected singularities (i.e. $r_0 = 0$) can be written as follows.

Source $\quad \mathscr{S} = C_0.$ (27*a*)

Dipoles $\quad \mathscr{D}_x = -ik\dfrac{\cos\Theta}{1-M_c\cos\Theta}C_0,$ (27*b*)

$$\mathscr{D}_y = C_1\cos\theta, \tag{27c}$$

$$\mathscr{D}_z = C_1\sin\theta. \tag{27d}$$

Quadrupoles

$$\mathscr{Q}_{xx} = -\frac{k^2\cos^2\Theta}{(1-M_c\cos\Theta)^2}C_0, \tag{27e}$$

$$\mathscr{Q}_{xy} = -ik\frac{\cos\Theta}{1-M_c\cos\Theta}\mathscr{D}_y, \tag{27f}$$

$$\mathscr{Q}_{xz} = -ik\frac{\cos\Theta}{1-M_c\cos\Theta}\mathscr{D}_z, \tag{27g}$$

$$\mathscr{Q}_{yy} = 2\,C_2\cos 2\theta + \tfrac{1}{2}(k^2f^2(0))\,C_0 - C_0M''(0)/(1-M_J\cos\Theta), \tag{27h}$$

$$\mathscr{Q}_{zz} = -2C_2\cos 2\theta + \tfrac{1}{2}(k^2f^2(0))\,C_0 - C_0M''(0)/(1-M_J\cos\Theta), \tag{27i}$$

$$\mathscr{Q}_{yz} = 2C_2\sin 2\theta. \tag{27j}$$

Here

$$f(0) = \frac{[\cos^2\Theta-(1-M_J\cos\Theta)^2]^{\frac{1}{2}}}{1-M_c\cos\Theta}, \tag{28k}$$

$$M''(0) = (d^2M/dr^2)_{r=0}. \tag{28l}$$

Note that Θ denotes the angle with respect to the jet axis, y and z are Cartesian co-ordinates in the transverse plane ($y = r\cos\theta$, $z = r\sin\theta$) and θ is the polar angle in our original cylindrical co-ordinate system.

In deriving (27) we used the fact that each time we differentiated† $\mathscr{S}$ with respect to x_0' we bring down a factor of $-ik\sigma_*$.

The above expressions for the dipoles and quadrupoles are valid within the zone of silence of round jets. Outside this zone, we set $r_\sigma \equiv 0$ and replace $f^2(0)$ by $-g^2(0;\sigma_*)$.

Using a familiar notation, we now write $\mathscr{Q}_{ij}$ for the ij quadrupole (for example $\mathscr{Q}_{12} = \mathscr{Q}_{xy}$) and define

$$a_{ij} = \langle|\mathscr{Q}_{ij}|^2\rangle\,(1-M_c\cos\Theta), \tag{29a}$$

where

$$\langle F\rangle = \frac{1}{2\pi}\int_0^{2\pi}F\,d\theta. \tag{29b}$$

Thus a_{ij} is the circumferential average of the amplitude of the ij quadrupole. The presence of the extra factor $1-M_c\cos\Theta$ in (29*a*) is thoroughly discussed by

† Here x_0' denotes the x' co-ordinate of the source. Since $\partial/\partial x_0' = -\partial/\partial x'$, we can set $x_0' = 0$ and differentiate with respect to x' rather than x_0'.

Ffowcs Williams (1963). According to Ribner (1969), the mean-square pressure of the self-noise component p^2_{SN} is given by

$$p^2_{SN} = \text{constant}\,[a_{11}+4a_{12}+2a_{22}+2a_{23}]. \tag{30}$$

We remark that (30) is a somewhat (but very slightly) simplified version of the Ribner result. The constant of proportionality appearing in (30) is a known function of the turbulent length scale and velocity fluctuations. More realistically, however, this constant should be regarded as an empirical quantity whose value is determined by matching the theoretical and experimental predictions at one angle, say $\Theta = 90°$. Thus (30) predicts the shape of the far-field pressure distribution but it does not give its absolute level.

Before we discuss the results of this theory it is perhaps worthwhile to review and summarize the most significant physical and mathematical assumptions. First, we have assumed that the pressure fluctuations in the jet obey Lilley's equation, whose right-hand side contains two forcing terms, one of which is generally called *self-noise* and the other *shear noise*. A detailed set of assumptions implicit in the derivation of Lilley's equation is given by Lilley (1972) and Mani (1976*a*).

In the Lilley formulation the self- and shear-noise forcing terms are qualitatively similar; they both are *quadratic* in the velocity fluctuations and both are essentially double divergences. The operator D_U acting on the self-noise is equivalent to multiplication by ω (the frequency of the eddy in its moving reference frame). The mean flow gradient dU/dr multiplying the shear-noise term is also proportional to ω, as may be seen from the experimental work of Davies, Fisher & Barratt (1963). On the other hand, the shear-noise quadrupole does exhibit a somewhat preferred axial orientation whereas the self-noise is isotropic. If this latter aspect of shear noise is ignored, the right-hand side of Lilley's equation can be *approximated* by a more or less isotropic quadrupole. This approximation is adopted in the present theory to facilitate theory/data comparison. Clearly this approximation is *not* necessary but it seems to be sufficiently accurate for the prediction of the noise of cold and hot jets (Mani 1976*a*, *b*).

The various quadrupole solutions $\mathcal{Q}_{ij}$ (or more precisely a_{ij}) are added with appropriate weighting factors such that the resultant sum represents an eddy of convected isotropic turbulence. These weighting factors come from Ribner's theory of self-noise (1969). Now we have stated that for the *Lilley formulation* the shear- and self-noise terms are extremely similar, so that we need not 'distinguish' between them and can regard them as an effective self-noise source. One may ask, then, why not regard them as an effective shear-noise source and use the weighting factors of Ribner's shear-noise theory? The answer to this question is very simple: whereas the *self-noise* terms of the Lilley and Lighthill formulations are extremely *similar*, the *shear-noise* terms are vastly *different*. Thus we have virtually no justification for using the Ribner shear-noise weighting factors in the present theory. For details of the assumptions made by Ribner we refer the interested reader to the cited work.

As in the Lighthill theory, we further assume that the quadrupoles are acoustically compact and are convecting with a constant subsonic convection Mach number M_c. As is customary, M_c is taken to be 65 % of the jet exit Mach number. The quadrupoles are assumed to lie on the jet axis.

Finally, the present theory is valid for the high frequency component of the noise of subsonic cold round jets.

Clearly not all of these assumptions can be justified rigorously! However, it seems, at least to us, that we have judiciously and simultaneously combined the most important ingredients of jet noise – quadrupole source generation,† eddy convection and acoustic/mean-flow interaction – into a rational structure that explains many of the observed characteristics of jet noise at intermediate to high frequencies.

7. Discussion

The most significant finding of this paper is that within the zone of silence (i.e. $0 < \Theta < \Theta_c = \cos^{-1}[(1+M_J)^{-1}]$) the sound pressure level SPL_{SN} due to self-noise is given by

$$\begin{aligned} SPL_{SN} &= 10\log_{10} p_{SN}^2 \\ &= -8\cdot 6858\left[ka\int_0^{r_\sigma/a} f\,d(r/a)\right] \\ &\quad + 10\log_{10}\left[\frac{\cos^4\Theta + 2\bar{f}^2\cos^2\Theta + \bar{f}^4}{(1-M_J\cos\Theta)^2(1-M_c\cos\Theta)^5}\right], \end{aligned} \tag{31a}$$

where
$$f(r) = \frac{\{\cos^2\Theta - [1-M(r)\cos\Theta]^2\}^{\frac{1}{2}}}{1-M_c\cos\Theta} \tag{31b}$$

and $\bar{f} = (1-M_c\cos\Theta)f(0)$. The first term in (31*a*) accounts for high frequency 'refraction'; the amount of refraction is clearly independent of the order of the singularity [see (26*a*)]. Thus the pressure fluctuations from simple sources, dipoles and quadrupoles, etc., are all refracted by the same amount, which is proportional to the frequency. For a given jet velocity profile $M(r)$ this integral can be readily evaluated for each location in the far field.

The second term in (31*a*) exhibits a convective amplification factor

$$(1-M_c\cos\Theta)^{-5}(1-M_J\cos\Theta)^{-2}.$$

Furthermore, this convection amplification factor is also present outside the zone of silence. When $M = M_J = 0$ (but $M_c \neq 0$) we recover the classical Ribner–Ffowcs Williams result; however, for non-zero M_J, it appears that the jet flow produces additional convective amplification. In fact, this convection factor is very similar to Mani's (1976*a*) *low* frequency result that accounts for the low 'frequency lift' at shallow angles (see also the experimental data of Lush 1971). In the calculations this additional amplification never really shows up at high frequencies because it is cancelled by the refractive effect for the most part. The results contained in (31*a*) clearly reveal the importance of acoustic/mean-flow

† Monopole- and dipole-source generation can be ignored in the present theory ($k \to \infty$) since they are of order k^0 and k^1 respectively whereas the quadrupole pressure is of order k^2.

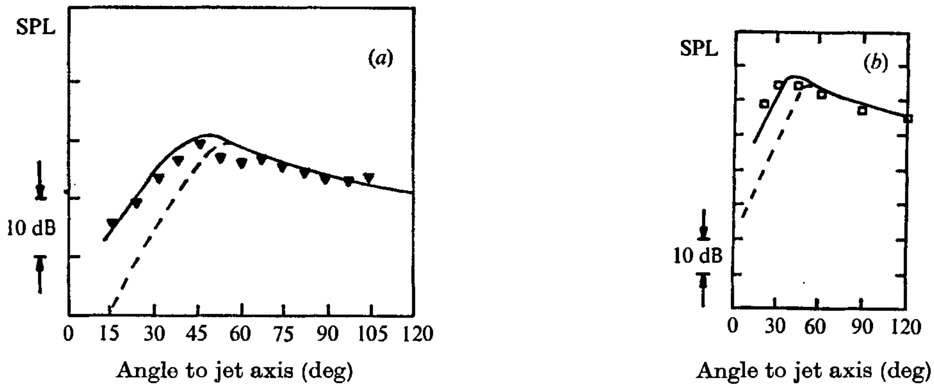

FIGURE 2. Comparison of theoretical and experimental results. Source Strouhal number = 1·0. (a) Jet velocity = 300 m/s. (b) Jet velocity = 1000 ft/s. ▼, experimental data of Lush (1971); □, experimental data of Ahuja & Bushell (1973); ----, theory of Mani (1976a); ——, present theory.

interactions. Nevertheless, these results also show that at high frequencies it is possible to separate convection and refraction completely, although the present convection factor differs somewhat from the classical result. Thus, using Lighthill's theory to account for convection and a separate theory to account for refraction would yield a result different from ours. Furthermore, the refraction integral is *independent* of the order of the singularity.

One additional point to note is that certain quadrupoles (such as $\mathscr{Q}_{yy}$) contain source-like terms whose coefficients are essentially the mean flow gradients. Similar terms play a fairly important role for hot jets at low frequencies (where they depend on the temperature gradient) but are generally ignored for cold jets (Mani 1976b). Thus in the theory/data comparison we drop the terms in (27 h, i) that are proportional to $M''(0)$ since they are of lower order in k.

We now come to the theory/data comparison. What we need to do is to relate M_c and M_J to those jet properties that are usually measured. As is customary, we set $M_c = 0{\cdot}65\,M_J^{(0)}$, where $M_J^{(0)}$ is the jet exit Mach number. The jet velocity profile is represented by

$$M(r) = M_J \exp\left[-\chi(r/a)^2\right], \tag{32}$$

where χ is a constant. To account very crudely for jet spread, we set $M_J = 0{\cdot}65 M_J^{(0)}$ then the conservation of momentum in the axial direction determines χ. In other words, momentum conservation yields one relationship between M_J and χ. In this respect, the present theory is semi-empirical with one adjustable constant (say M_J) outstanding. Since the mean flow profiles of our theory have no axial variation, some kind of average value for M_J must be chosen. Our choice seems reasonable but is by no means unique.

The theoretical predictions are shown in figures 2–4, where the *levels* of the theoretical curves have been adjusted to obtain good agreement with the experimental data at $\Theta \approx 90°$. The agreement between the present theory and various experimental data is good when the source Strouhal number $\{= ka/\pi M_J)$ is about unity but, surprisingly enough, not very good at a source Strouhal

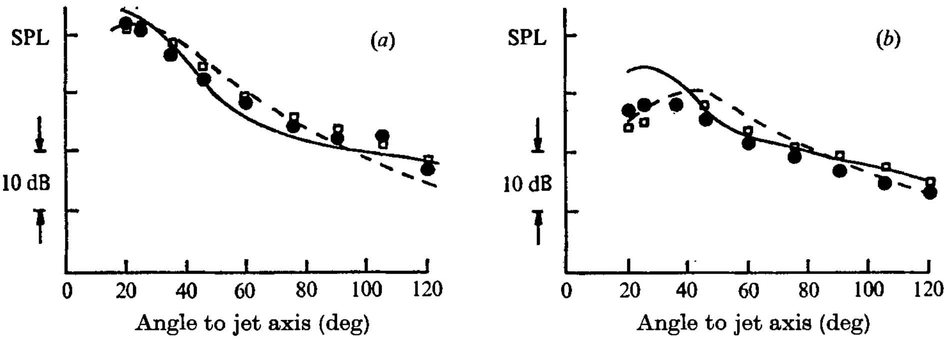

FIGURE 3. Comparison of theoretical and experimental results. Jet velocity = 986·5 ft/s. (*a*) Source Strouhal number = 0·136. (*b*) Source Strouhal number = 0·346. □, ●, experimental data of Olsen (private communication); ----, theory of Mani (1976*a*); ———, present theory.

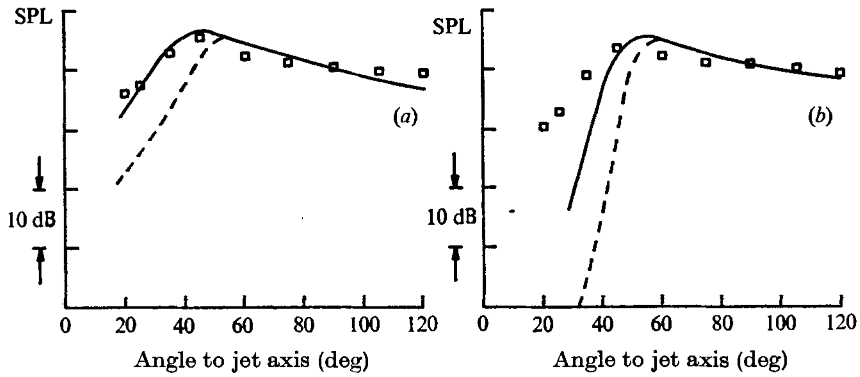

FIGURE 4. Comparison of theoretical and experimental results. Jet velocity = 986·5 ft/s. (*a*) Source Strouhal number = 1·0. (*b*) Source Strouhal number = 2·0. □, experimental data of Olsen (private communication); ----, theory of Mani (1976*a*); ———, present theory.

number of two at shallow angles. The corresponding slug-flow predictions of Mani are also shown in these figures. Thus we find that the refractive effects are still overestimated even though there is a 10–15 dB improvement over Mani's results.

These calculations clearly show the inadequacy of our theory at very high frequencies. At these frequencies, jet spread and turbulent scattering are *very important* and must be incorporated in every theory to obtain good quantitative agreement. Nevertheless, it now appears that we are in a position to predict† the directivity of jet noise up to source Strouhal numbers somewhat greater than unity. Interestingly enough, both Mani's and the present results shown in figures 2 and 4 include only the self-noise component of jet noise. This probably implies that the self- and shear-noise components (based on the Lilley formulation) are so similar that one really does not need to distinguish between them.

Figure 3 shows that the present theory is not too bad even at fairly low Strouhal numbers. This is, of course, fortuitous but indicates the range of applicability of

† Of course, at low frequencies, Mani's theory gives very good results.

the theory. We remark that by using Lilley's equation we have introduced another singularity when $M_J \cos \Theta = 1$. We do not know at this point how to 'remove' this singularity, although our theory is valid for $M_J > 1$ as long as $M_c < 1$ and $M_J - M_c < 1$.

Finally we wish to point out that the present theory combines the classical ideas of Lighthill, Ribner and Ffowcs Williams with those of Mani to provide a simple result for the estimation of acoustic/mean-flow interaction in jet noise. Extension of this work to hot jets and to off-axis singularities is currently in progress.

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